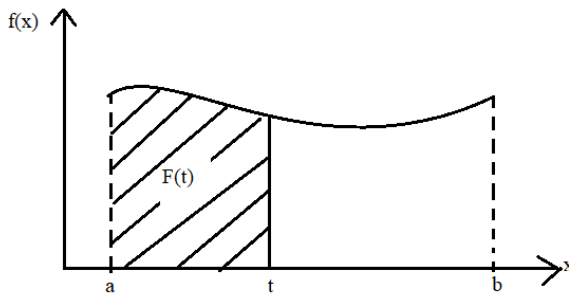


CUMULATIVE DISTRIBUTION FUNCTION, $F(x)$

If X is a continuous random variable with p.d.f, $f(x)$, cumulative distribution function, $F(x)$ for a value of t in the range of the function is given by;

$$F(t) = P(X \leq t) = \int_{-\infty}^t f(x)dx$$

In practice, the lower limit, $-\infty$ is the smallest possible value of x in the range for which x is valid. So if $f(x)$ is valid in the range $a \leq x \leq b$, then $F(t) = \int_a^t f(x)dx$



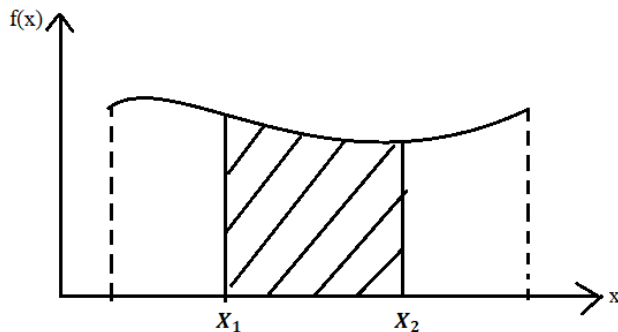
Note: $F(t)$ gives the area under the curve, $f(x)$ up to a particular value t

$$F(b) = P(X \leq b)$$

$$= \int_a^b f(x)dx$$

$$= 1$$

Finding $P(X_1 \leq X \leq X_2)$ using $F(x)$



$$P(X_1 \leq X \leq X_2) = F(X_2) - F(X_1)$$

Finding the median, Quartiles and Percentiles

Median is the value 50% of the way through the distribution. If m is the median, then for $f(x)$ defined for $a \leq x \leq b$,

$$\int_a^m f(x)dx = 0.5 \rightarrow F(m) = 0.5$$

The lower quartile, q_1 is the value 25% of the way through the distribution

$$\int_a^{q_1} f(x)dx = 0.25 \rightarrow F(q_1) = 0.25$$

The lower quartile, q_3 is the value 75% of the way through the distribution

$$\int_a^{q_3} f(x)dx = 0.75 \rightarrow F(q_3) = 0.75$$

Percentiles divide a set of distribution into 100 equal parts

Hence $F(n^{th} \text{ percentile}) = \frac{n}{100}$ e.g $F(10^{th}) = 0.1$, $F(70^{th}) = 0.7$, etc

Examples

1. The continuous random variable X has a p.d.f given by;

$$f(x) = \begin{cases} k(4 - x^2), & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Where k is a constant

- (a) Show that $k = \frac{3}{16}$
- (b) Find the cumulative distribution function, $F(x)$
- (c) Determine $P(0.6 < X < 1.8)$

Solution

$$(a) \quad \int_0^2 k(4 - x^2)dx = 1,$$

$$k \left[4x - \frac{x^3}{3} \right]_0^2 = 1$$

$$k \left[\left(4x2 - \frac{2^3}{3} \right) - 0 \right] = 1 \quad \therefore k = \frac{3}{16}$$

$$(b) \quad F(t) = \int_0^t \frac{3}{16} (4 - x^2)dx = \frac{3}{16} \left[4x - \frac{x^3}{3} \right]_0^t = \frac{3}{16} \left[\left(4t - \frac{t^3}{3} \right) - \left(4x0 - \frac{0^3}{3} \right) \right]$$

$$= \frac{3}{16} \left(4t - \frac{t^3}{3} \right)$$

$$\text{Hence } F(x) = \frac{3}{16} \left(4x - \frac{x^3}{3} \right)$$

$$(c) \quad P(0.6 < X < 1.8) = F(1.8) - F(0.6) = \frac{3}{16} \left[\left(4x1.8 - \frac{1.8^3}{3} \right) - \left(4x0.6 - \frac{0.6^3}{3} \right) \right]$$

$$= 0.5490$$

2. The probability density function of a continuous random variable x is given by;

$$f(x) = \begin{cases} \frac{2}{13}(x+1), & 0 \leq x \leq 2 \\ \frac{2}{13}(5-x), & 2 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

- (i) Find the cumulative function and hence, sketch $F(x)$
- (ii) Calculate the median and 20th percentile
- (iii) Find $P(x < 2.5)$

Solution

- (i) In the range $0 \leq x \leq 2$

$$F(t) = \int_0^t \frac{2}{13}(x+1)dx = \frac{2}{13} \left[\frac{x^2}{2} + x \right]_0^t = \frac{2}{13} \left[\left(\frac{t^2}{2} + t \right) - 0 \right] = \frac{2}{13} \left(\frac{t^2}{2} + t \right)$$

This implies; $F(x) = \frac{2}{13} \left(\frac{x^2}{2} + x \right) = \frac{1}{13} (x^2 + 2x)$

$$F(2) = \frac{1}{13} (4 + 4) = \frac{8}{13}$$

In the range $2 \leq x \leq 3$

$$\begin{aligned} F(t) &= F(2) + \int_2^t \frac{2}{13}(5-x)dx = \frac{8}{13} + \frac{2}{13} \left[5x - \frac{x^2}{2} \right]_2^t \\ &= \frac{8}{13} + \frac{2}{13} \left[\left(5t - \frac{t^2}{2} \right) - \left(5 \cdot 2 - \frac{2^2}{2} \right) \right] \\ &= \frac{8}{13} + \frac{2}{13} \left(5t - \frac{t^2}{2} - 8 \right) = \frac{1}{13} (10t - t^2 - 8) \end{aligned}$$

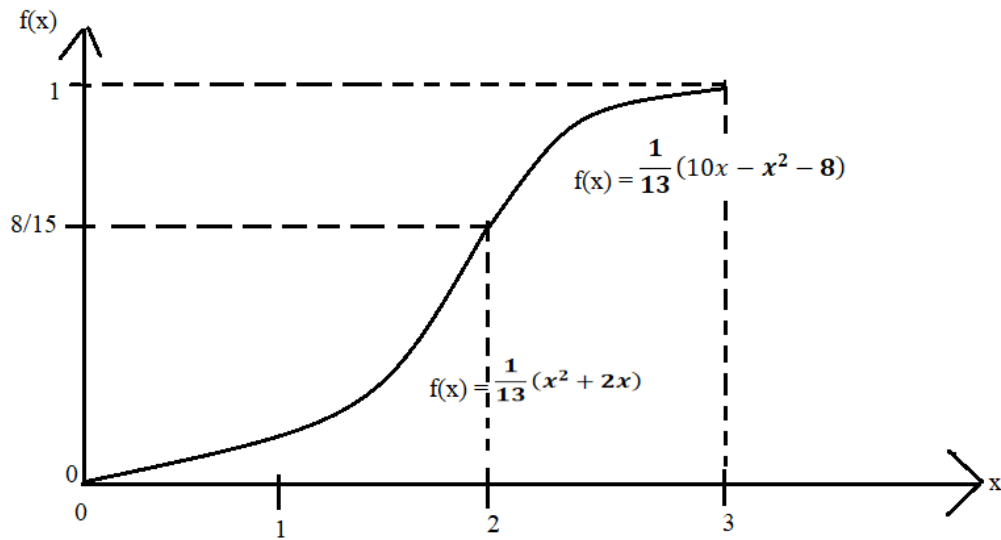
This implies; $F(x) = \frac{1}{13} (10x - x^2 - 8)$

$$F(3) = \frac{1}{13} (10 \cdot 3 - 3^2 - 8) = 1$$

Hence;

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{13} (x^2 + 2x); & 0 \leq x \leq 2 \\ \frac{1}{13} (10x - x^2 - 8); & 2 \leq x \leq 3 \\ 1; & x \geq 3 \end{cases}$$

Sketch of F(x)



(ii) Since $F(2) = \frac{8}{15} > 0.5$ then $F(m) = 0.5$

$$\rightarrow \frac{1}{13}(m^2 + 2m) = 0.5$$

$$\rightarrow m^2 + 2m - 6.5 = 0$$

$$\rightarrow m = \frac{-2 \pm \sqrt{2^2 - 4(1)(-6.5)}}{2(1)} = 1.7386 \text{ or } -3.7386$$

Hence median = 1.7386

20th percentile, $F(P_{20}) = 0.2$

$$\frac{1}{13}(P_{20}^2 + 2(P_{20})) = 0.2$$

$$P_{20}^2 + 2(P_{20}) - 2.6 = 0$$

$$P_{20} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2.6)}}{2(1)} = 0.8974 \text{ or } -2.8974$$

Hence $P_{20} = 0.8974$

(iii) $P(x < 2.5) = F(2.5) = \frac{1}{13}(10(2.5) - 2.5^2 - 8) = 0.8269$

3. The continuous random variable X has cumulative distribution function F(x).

$$F(x) = \begin{cases} 0, & x \leq 1 \\ \frac{(x-1)^2}{12} & ; 1 \leq x \leq 3 \\ \frac{14x - x^2 - 25}{24} & ; 3 \leq x \leq 7 \\ 1 & ; x \geq 7 \end{cases}$$

Find the;

- (a) median of X
- (b) $P(x > 2.5 / x < 4)$
- (c) interquartile range

Solution

(a) Let m be the median.

Testing for $F(m) \geq 0.5$

$$F(3) = \frac{(3-1)^2}{12} = 0.3333 < 0.5$$

Hence the median is in the range $1 \leq x \leq 3$

$$F(m) = \frac{14m - m^2 - 25}{24} = 0.5$$

$$14m - m^2 - 25 = 12$$

$$m^2 - 14m + 37 = 0$$

Solving the quadratic equation yields; $m = 10.4641$ or 3.5359

$$(b) P(x > 2.5 / x < 4) = \frac{P(x > 2.5 \text{ and } x < 4)}{P(x < 4)} = \frac{P(2.5 < x < 4)}{P(x < 4)} = \frac{F(4) - F(2.5)}{F(4)}$$

$$= \frac{\frac{14 \times 4 - 4^2 - 25}{24} - \frac{(2.5-1)^2}{12}}{\frac{14 \times 4 - 4^2 - 25}{24}} = 0.7$$

(c) Let q_1 and q_3 be the lower and upper quartiles respectively

$$F(q_1) = 0.25$$

$$\frac{(q_1-1)^2}{12} = 0.25 \rightarrow (q_1 - 1)^2 = 3 \quad \therefore q_1 = 2.7321$$

$$F(q_3) = 0.75$$

$$\frac{14q_3 - (q_3)^2 - 25}{24} = 0.75 \quad \rightarrow (q_3)^2 - 14q_3 + 43 = 0$$

$$\rightarrow q_3 = 9.4495 \text{ or } 4.5505$$

$$\therefore q_3 = 4.5505$$

$$\text{Hence Interquartile range} = (q_3 - q_1) = (4.5505 - 2.7321) = 1.8184$$

Trial questions

1. The continuous random variable X has a pdf given by;

$$f(x) = \begin{cases} cx^2; & 0 \leq x \leq 2 \\ 2c(4-x) & ; 2 \leq x \leq 4 \\ 0; & \text{otherwise} \end{cases}$$

where c is a constant

Find the;

- (a) value of c
- (b) lower quartile
- (c) $P(1.5 < x < 2.7)$

2. X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} 2kx; & 0 \leq x \leq 1 \\ k(x^2 - 4x + 5); & 1 \leq x \leq 3 \\ 2k(4-x); & 3 \leq x \leq 4 \\ 0; & \text{elsewhere.} \end{cases}$$

- (a) Find the value of the constant k
- (b) Find F(x), the cdf of X, hence compute $P(|X - 2| < 1.5)$

3. The probability density function of a random variable X is given by;

$$f(x) = \begin{cases} kx^2 & 0.1 \leq x \leq 0.25 \\ \frac{1}{4}k(0.5-x) & 0.25 \leq x \leq 0.5 \\ 0 & \text{Otherwise} \end{cases}$$

Find the;

- (a) value of k and hence sketch the function F(x)
- (b) 30th percentile and semi interquartile range of X

4. A random variable X has the cumulative distribution function given below.

$$F(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x^2}{16} & ; 0 \leq x \leq 2 \\ ax - b & ; 2 \leq x \leq 4 \\ \frac{3}{4}x - \frac{1}{16}x^2 - \frac{5}{4} & ; 4 \leq x \leq 6 \\ 1 & 6 < x \end{cases}$$

Find (i) the value of the constants a and b

(ii) $P(1 \leq x \leq 5/x \geq 2)$

5. The probability density function of a random variable X is given by;

$$f(x) = \begin{cases} kx; & 0 < x < 2 \\ 2k(x-1)^2 & ; 2 < x < 5 \\ 0; & elsewhere \end{cases}$$

Find the;

- (a) value of constant k
- (b) cumulative function, F(x)
- (c) $P(|x-3| < 1)$

Finding p.d.f, f(x) from Cumulative function F(x)

Since F(x) can be got by integrating f(x), then f(x) can be obtained by differentiating F(x)

$$f(x) = \frac{d}{dx}(F(x)) = F'(x)$$

Examples

1. The continuous random variable X has a cumulative distribution function F(x) given by;

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2x}{3} & 0 \leq x \leq 1 \\ \frac{x}{3} + k & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

Find the;

- (a) value of k
- (b) p.d.f, $f(x)$ and sketch it

Solution

- (a) At $x = 1$

$$\frac{2(1)}{3} = \frac{1}{3} + k \quad \therefore k = \frac{1}{3}$$

- (b) In the range $0 \leq x \leq 1$

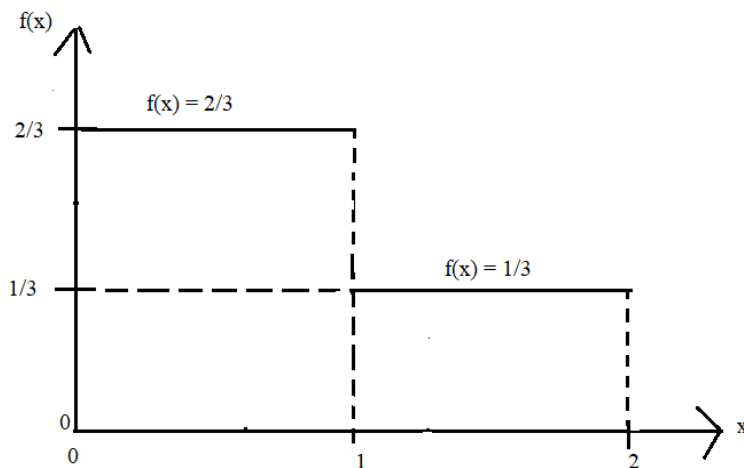
$$f(x) = \frac{d}{dx} \left(\frac{2x}{3} \right) = \frac{2}{3}$$

- In the range $1 \leq x \leq 2$

$$f(x) = \frac{d}{dx} \left(\frac{x}{3} + \frac{1}{3} \right) = \frac{1}{3}$$

$$\text{Hence } f(x) = \begin{cases} \frac{2}{3}, & 0 \leq x \leq 1 \\ \frac{1}{3}, & 1 \leq x \leq 2 \\ 0, & \text{else where} \end{cases}$$

Sketch of $f(x)$



2. The continuous random variable X has cumulative distribution, $F(x)$ is given by;

$$F(x) = \begin{cases} 0; & x \leq 1 \\ \frac{(x-1)^2}{12}; & 1 \leq x \leq 3 \\ \frac{(14x - x^2 - 25)}{24}; & 3 \leq x \leq 7 \\ 1; & x \geq 7 \end{cases}$$

Find the;

- (a) p.d.f, $f(x)$ and sketch it
- (b) $E(X)$
- (c) $P(2.8 < X < 5.2)$

Solution

(a) In the range: $1 \leq x \leq 3$;

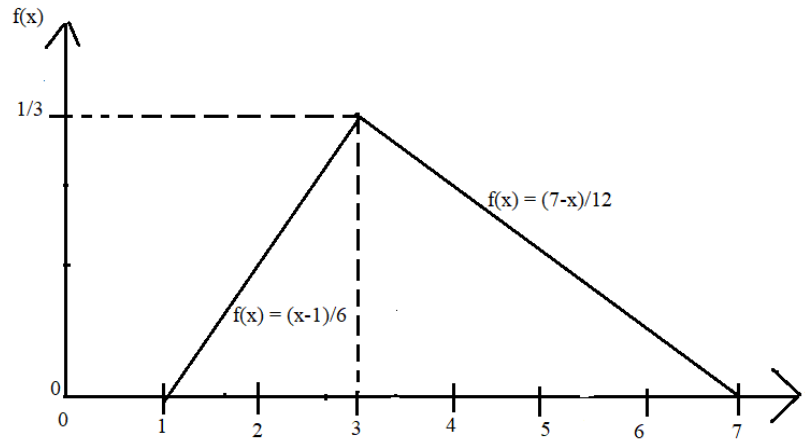
In the range: $3 \leq x \leq 7$;

$$f(x) = \frac{d}{dx} \left(\frac{x-1}{12} \right)^2 = \frac{x-1}{6}$$

$$f(x) = \frac{d}{dx} \left(\frac{14x - x^2 - 25}{24} \right) = \frac{7-x}{12}$$

$$\text{Hence } f(x) = \begin{cases} \frac{x-1}{6}; & 1 \leq x \leq 3 \\ \frac{7-x}{12}; & 3 \leq x \leq 7 \\ 0; & \text{otherwise} \end{cases}$$

Sketch of f(x)



$$\begin{aligned} \text{(b)} E(x) &= \int_1^3 x \left(\frac{x-1}{6} \right) dx + \int_3^7 x \left(\frac{7-x}{12} \right) dx \\ &= \frac{1}{6} \int_1^3 (x^2 - x) dx + \frac{1}{12} \int_3^7 (7x - x^2) dx \\ &= \frac{1}{6} \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^3 + \frac{1}{12} \left[\frac{7x^2}{2} - \frac{x^3}{3} \right]_3^7 \\ &= \frac{1}{6} \left[\left(9 - \frac{9}{2} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right] + \frac{1}{12} \left[\left(\frac{343}{2} - \frac{343}{3} \right) - \left(\frac{63}{2} - 9 \right) \right] \\ &= \frac{7}{9} + \frac{26}{9} = 3\frac{2}{3} \text{ or } 3.6667 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } P(2.8 < X < 5.2) &= F(5.2) - F(2.8) \\
 &= \frac{14(5.2) - (5.2)^2 - 25}{24} - \frac{(2.8 - 1)^2}{12} \\
 &= \mathbf{0.5950}
 \end{aligned}$$

Trial questions

1. A random variable X has a cumulative distribution function given below.

$$F(x) = \begin{cases} 0, & x \leq 0 \\ ax, & 0 \leq x \leq 1 \\ \frac{x+b}{3}, & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

Find;

- the value of a and b
 - $P(X < 1.5/X > 1)$
 - Mean of X
2. The cumulative function of a random variable X is given by;

$$F(x) = \begin{cases} 0; & x < -1 \\ \beta(x + 1); & -1 \leq x \leq 0 \\ \beta(2x + 1); & 0 \leq x \leq 1 \\ 3\beta; & 1 \leq x \end{cases}$$

Determine the;

- Value of β
 - P.d.f, $f(x)$ of X
 - Standard deviation, σ of X
 - $P(|X - \mu|) = \frac{1}{3}$
3. A random variable X has a Cumulative distribution function, $F(x)$ given by;

$$F(x) = \begin{cases} 0; & x < 0 \\ \frac{x^2}{2}; & 0 \leq x \leq 1 \\ \left(2x - \frac{x^2}{2} - 1\right); & 1 \leq x \leq 2 \\ 1; & x \geq 2 \end{cases}$$

Find the;

- p.d.f, $f(x)$ and sketch it
- variance of X
- $P(0.5 < X < 1.3/X > 1)$